A Mismatched Decoding Perspective of Channel Output Quantization

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***Abstract*—Channel output quantization to a smaller number of outputs is modeled as a mismatched decoding problem. The conditions that a mismatched decoding metric should satisfy in order to represent an output quantizer are derived. In addition, a mismatched decoding metric and hypothesis test that minimizes the average error probability are found. It is shown that the best possible mismatched decoder is equivalent to maximum-likelihood decoding for the channel between the channel input and the quantized output. This gives a class of mismatched decoding problems where the mismatch capacity is known. This result supports previous studies on quantizer design and optimization over the quantized channel.**

I. Introduction

One of the key issues in hardware implementation of communication systems is quantization of the received channel values, needed prior to any subsequent processing. In such implementations, there is a trade-off between hardware complexity (represented by the number of quantization levels) and the error performance of the system. It is thus of interest to use as few quantization levels as possible while maintaining a desired error performance for a given transmission rate. Apart from channel quantization, there are other applications where quantization is relevant such as the implementation of message-passing decoders [1] and the construction of polar codes [2].

Quantization has been extensively studied in the literature. Most studies have focussed on designing the quantizer based on a given performance metric for the combined channel from the input X to the quantizer output Z referred to as quantized channel (see Fig. 1). Multiple design criteria related to the quantized channel have been considered such as the cut-off rate [3], [4], mutual information [5]–[8] or error exponent [9].

A recent example is [8] where maximum mutual information quantizer design for discrete memoryless channels is considered. It is shown that there is a deterministic quantizer that maximizes the mutual information of the quantized channel. In addition, utilizing results from learning theory [10], a separating hyperplane condition for the optimality of the quantizer is obtained. For the special case of the binary-input channels, an algorithm based on dynamic programming is developed to find an optimal quantizer. For the non-binary input case, different suboptimal approaches with polynomial complexities have been proposed in the literature including [1], [2], [13].

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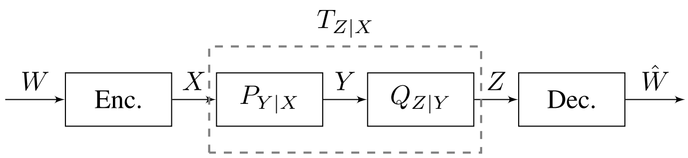

Fig. 1: Coded communication over a discrete channel followed by a quantizer and matched decoder to the quantized channel.

The problem of decoding when the metrics are constrained to take on values from a finite set (integrer values) are investigated in [11], [12], which is motivated by practical applications where the received signals and soft decoding metrics need to be quantized. Generalized cut-off rate [11] and mismatch capacity [12] are used as the performance measures and the integer metric assignments which maximize these criteria are designed. Except these few studies, most of the previous literature consider the channel between the input and the quantized output, and optimize the quantizer according to a performance metric related to the quantized channel.

We find the mismatch decoding approach more natural, and assume the quantizer as part of the decoder. In this case, the decoder performs possibly suboptimal decoding due to implementation constraints. This is precisely the definition of mismatched decoding problem [14], [15]. The question under investigation is whether there is anything to be gained by using mismatch decoding approach or both approaches are equivalent.

The main contribution of this paper is to study channel output quantization from a mismatched decoding perspective. We first derive conditions that the mismatched decoding metric needs to satisfy in order to represent the channel quantizer. Then, we derive a lower bound on the probability of error where we find an optimal mismatched metric and the corresponding test at the receiver. We show that the best mismatched decoding coincides with maximum likelihood decoding for the quantized channel, hence giving another example of mismatched decoding where the mismatch capacity is known. Our findings thus support previous results that consider quantizer optimization over the quantized channel.

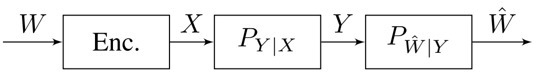

Fig. 2: Coded communication over a discrete channel followed by a mismatched decoder.

II. Problem Set-up

We consider *one-shot* communication over a discrete channel followed by a quantizer at the output, as shown in Fig. 1. The message w, drawn equiprobably from the message set $\mathcal{W} = \{1, 2, \ldots, M\}$, is encoded onto $X_w \in \mathcal{X}$ which is received by the receiver as a random observation $Y \sim P_{Y|X}(y|x)$. The encoder and decoder share a codebook $\mathcal{C} = \{X_1, \ldots, X_M\}$. The channel input and output alphabets are $\mathcal{X}$ and $\mathcal{Y}$ of cardinality $|\mathcal{X}|$ and $|\mathcal{Y}|$, respectively. At the receiver side, a (possibly random) quantizer produces Z from Y; Z takes on values from finite alphabet $\mathcal{Z}$ with cardinality $|\mathcal{Z}| = K$. The quantizer is defined by set of probabilities $Q_{Z|Y}(z|y)$. The conditional probability of the quantizer output given the channel input can be obtained as

$$T_{Z|X}(z|x) = \sum_{y \in \mathcal{Y}} Q_{Z|Y}(z|y) P_{Y|X}(y|x). \quad (1)$$

The set-up considered here is general and includes the channels and quantizers with memory as well as standard memoryless models. The majority of the literature considers a specific form of the quantized channel as a combination of the memoryless channel with single use transition probability $P_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x})$ with a single channel use quantizer $Q_{\mathsf{Z}|\mathsf{Y}}(\mathsf{z}|\mathsf{y})$ (We use $\mathsf{x}, \mathsf{y}, \mathsf{z}$ to denote single letter variables to differentiate from one-shot notation). Specifically, they search for the quantizer Q^* which is the solution of the following optimization problem [5]–[8],

$$Q^* = \arg\max_{Q \in \mathcal{Q}} I(\mathsf{X}; \mathsf{Z}) \quad (2)$$

where $\mathcal{Q}$ is the set of all possible quantizers. One-shot communication problem considered here includes this special case.

We consider the quantizer as part the decoder (as shown in Fig. 2) and assume that the receiver is designed to perform suboptimal decoding. At the receiver, given the channel output y, an estimate of the transmitted message is formed based on some function $q(w, y)$ which is called the decoding metric (for a fixed encoder with one-to-one mapping $q(w, y) = q(x, y)$ as in the standard form, although $q(w, y)$ is more general). A common way to obtain message estimate is using maximum metric decoding as follows,

$$\hat{w} = \arg\max_{w' \in \mathcal{W}} q(w', y). \quad (3)$$

When the decoding metric is not equivalent to that of the optimal maximum likelihood decoder $P_{Y|W}$, in the sense of yielding an identical decision rule, it is said that the decoder is mismatched [14], [15]. The decoder makes an error when its estimated message is different from the one sent, i.e., $\hat{w} \neq w$.

The mismatched decoding problem in its classical form studies reliable communication over a given channel with a given (possibly suboptimal) decoding rule. However, as we will see in this paper, the quantizer imposes some restrictions on the decoding metric.

III. Mismatched Decoding

A. Deterministic Quantizer

First we consider a deterministic quantization of the channel outputs, i.e.,

$$Q_{Z|Y}(z|y) \in \{0, 1\}, \text{ for all } z \in \mathcal{Z}, y \in \mathcal{Y}, \quad (4)$$

and derive a condition for the corresponding mismatched decoding rule. Such a deterministic quantizer $Q_{Z|Y}$ is a mapping function $Q_{Z|Y} : \mathcal{Y} \to \mathcal{Z}$ that partitions the set $\mathcal{Y}$ to K subsets $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_K\}$ such that $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset$ and $\bigcup_{k=1}^{K} \mathcal{Y}_k = \mathcal{Y}$ and labels each partition $\mathcal{Y}_k$ with new symbol z_k. In other words, all the channel outputs $y \in \mathcal{Y}_k$ are merged to the quantizer output z_k, for all $1 \leq k \leq K$.

Any decoding metric that does not discriminate between the merged outputs represents a deterministic quantizer and vice versa. This condition is stated concisely as follows.

Condition 1: A decoding metric represents a deterministic quantizer $Q_{Z|Y}$ (with K quantized outputs) if and only if it satisfies $q(w, y) = q_k(w)$ for all $y \in \mathcal{Y}_k$, where the functions $q_k(w)$ have the property of $q_k(w) \neq c q_j(w)$ for any constant c and $k \neq j$.

Hence, overall there are K decoding metrics, $q_k(w)$ $1 \leq k \leq K$, and each metric function is used for all the channel outputs in its corresponding subset $\mathcal{Y}_k$.

This condition does not specify any property for the decoding metric other than the merged outputs sharing the same metric. Therefore, the mismatched decoding problem under consideration is different from its classical form where the decoding metric is fixed but at the same time can be more general.

B. Randomized Quantizer

We now explore the properties that the mismatched decoding rule needs to satisfy in order to represent a general randomized quantizer $Q_{Z|Y}(z|y)$. We first split the randomized quantizer into two parts. The first part is a randomized expansion of the channel outputs as follows

$$y_i \to \bar{y}_{ij} \text{ w.p. } P_{\bar{Y}|Y}(\bar{y}_{ij}|y_i) = Q_{Z|Y}(z_j|y_i) \text{ for all } 1 \leq j \leq K \quad (5)$$

and the second part is a deterministic quantizer as

$$P_{Z|\bar{Y}}(z_k|\bar{y}_{ij}) = \mathbb{1}\{k = j\} \quad (6)$$

where $\mathbb{1}\{.\}$ is the indicator function. These variables form a Markov chain as $X - Y - \bar{Y} - Z$.

In order to illustrate this two-step representation, we consider an example of a discrete channel with binary input x and ternary output y and a randomized quantization to binary labels z. Fig. 3 shows the randomized quantizer and its counterpart as a combination of a randomized expansion and a deterministic quantizer for this example.

The randomized expansion of channel outputs to the expanded outputs $\bar{y}_{ij}$ does not change the amount of information

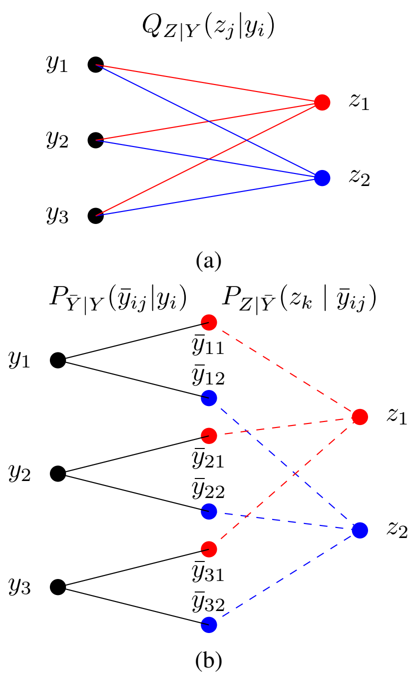

Fig. 3: (a) Randomized quantization from ternary outputs to binary labels. (b) Equivalent model with randomized expansion and deterministic quantizer.

about x, since the expansion probabilities are independent of x and hence decoding using y or $\bar{y}_{ij}$ is equivalent. Following Condition 1, a mismatched decoder with a decoding metric

$$q(w, \bar{y}_{ij}) = q_k(w) \text{ for all } 1 \leq i \leq |\mathcal{Y}|, j = k \tag{7}$$

represents a deterministic quantizer that merges all the virtual outputs $\bar{y}_{ij}$ with $j = k$ since it has the property of using the same decoding metric for all the merged virtual outputs. Therefore, the mismatched decoding metric for the original channel outputs y satisfies the following condition.

Condition 2: A decoding metric represents a randomized quantizer $Q_{Z|Y}$ if and only if it fulfills the following property

$$q(w, y) = q_k(w) \quad \text{w.p. } Q_{Z|Y}(z_k|y) \text{ for all } 1 \leq k \leq K. \tag{8}$$

In other words, for each channel output y, the decoding metric $q_k(w)$ is used with probability $Q(z_k|y)$. Note that Equation (8) includes the deterministic quantizer as a special case.

IV. Error Probability

The decoder estimates the transmitted message as $\hat{w}$ given the channel output y. It performs an M-ary hypothesis test based on the mismatched decoding metric $q(w, y)$ (8). The conditional distribution[1] describing the decoder output has the following form

$$P_{\hat{W}|Y}(w|y) = \sum_{k=1}^{K} P_{\hat{W}}^{k}(w) Q_{Z|Y}(z_k|y), \tag{9}$$

[1]For the memoryless channel case with single channel use quantizer and product form decoding metric of $q(w, \boldsymbol{y}) = \prod_{i=1}^{n} q(\mathsf{x}_i, \mathsf{y}_i)$, the conditional distribution of (9) is $P_{\hat{\boldsymbol{X}}|\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{y}) = \prod_{i=1}^{n} \sum_{k=1}^{K} P_{\hat{\mathsf{X}}}^{k}(\mathsf{x}_i) Q_{\mathsf{Z}|\mathsf{Y}}(\mathsf{z}_k|\mathsf{y}_i)$.

where $P_{\hat{W}}^{k}(w)$ is the probability that the decoder returns the estimate w using the decoding metric function $q_k(w)$, hence $\sum_{w \in \mathcal{W}} P_{\hat{W}}^{k}(w) = 1$. We denote the average error probability of the decoder by $\epsilon(P_{\hat{W}|Y})$. This probability is given by

$$\epsilon(P_{\hat{W}|Y}) \triangleq \mathbb{P}\left[\hat{W} \neq W\right] \tag{10}$$

$$= 1 - \sum_{w,y} P_{WY}(w, y) P_{\hat{W}|Y}(w|y). \tag{11}$$

Substituting (9) in (11) gives

$$\epsilon(P_{\hat{W}|Y}) = 1 - \sum_{w,y} P_{WY}(w, y) \Big(\sum_{k=1}^{K} P_{\hat{W}}^{k}(w) Q_{Z|Y}(z_k|y) \Big) \tag{12}$$

$$= 1 - \sum_{w} \sum_{k=1}^{K} P_{\hat{W}}^{k}(w) \Big(\sum_{y} P_{WY}(w, y) Q_{Z|Y}(z_k|y) \Big) \tag{13}$$

$$= 1 - \sum_{w} \sum_{k=1}^{K} P_{\hat{W}}^{k}(w) P_{WZ}(w, z_k). \tag{14}$$

Since $\sum_{w \in \mathcal{W}} P_{\hat{W}}^{k}(w) = 1$, the error probability in (14) can be bounded as

$$\epsilon(P_{\hat{W}|Y}) \geq 1 - \sum_{k=1}^{K} \max_{w'} P_{WZ}(w', z_k) \tag{15}$$

$$= 1 - \sum_{w,z} P_{WZ}(w, z) P_{\hat{W}|Z}^{\text{MAP}}(w|z). \tag{16}$$

The equality in (15) is achieved by using a maximum metric test with mismatched metric (8) where metric functions are given by

$$q_k(w) = \sum_{y \in |\mathcal{Y}|} P_{WY}(w, y) Q_{Z|Y}(z_k|y) = P_{WZ}(w, z_k). \tag{17}$$

The maximum metric test is described as

$$P_{\hat{W}}^{k}(w) = \begin{cases} \frac{1}{|\mathcal{S}_k|}, & \text{if } w \in \mathcal{S}_k \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where the set $\mathcal{S}_k$ is defined as $\mathcal{S}_k = \Big\{ w \,|\, P_{WZ}(w, z_k) = \max_{w'} P_{WZ}(w', z_k) \Big\}$. The analysis shows that the best mismatched decoding which is based on utilizing metrics as (8) and (17) is equivalent to the matched decoding for the quantized channel. Furthermore, it shows that there is inevitable loss due to the mismatch decoding and provides a way for calculating the mismatch capacity.

V. Discussion

The above error probability analysis shows the equivalence of the best mismatched decoder for the quantization problem (using the metrics given in (8) and (17)) to matched decoding over the quantized channel. It demonstrates that there is an inevitable loss due to the quantization that can not be compensated for with any encoding and decoding. This, in turn, gives an example of mismatch decoding where the mismatch capacity is known. The derivations in Section IV show the equivalence of this problem to the matched decoding over quantized channel and provides a way to calculate the mismatch capacity. In addition, this supports previous results that consider the quantized channel and design the quantizer in order to optimize the communication rate over that channel.

A natural question to ask is whether there are other instances of the mismatched decoding problem that would fall into the same category and result in a trivial solution for the mismatch capacity. Although we do not have the answer to this question we study another example in the following that is an interesting application of the derived result here.

Let us consider communication over a discrete memoryless channel (DMC). For an equiprobably chosen message w from the set $\mathcal{W} = \{1, 2, \dots, M\}$, the encoder maps it to a codeword of length n denoted by $\boldsymbol{x}(w) \in \mathcal{X}^n$. The corresponding channel output sequence $\boldsymbol{y}$ is generated according to $P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i)$. We consider the suboptimal decoder which selects the message maximizing the decoding metric $q(\boldsymbol{x}(w), \boldsymbol{y})$, i.e.,

$$\hat{w} = \arg\max_{w} q(\boldsymbol{x}(w), \boldsymbol{y}), \tag{19}$$

where the metric $q(\boldsymbol{x}(w), \boldsymbol{y})$ depends only on first $(n-1)$ outputs ignoring the last output, i.e., $q(\boldsymbol{x}(w), \boldsymbol{y}) = q(\boldsymbol{x}(w), y_1^{n-1})$ where y_1^{n-1} denotes the sequence of first $n-1$ indexes of output $\boldsymbol{y}$. This can be interpreted as an instance of quantization where all the output sequences that share the same first $n-1$ symbols and only differ in the last index are merged to the same quantizer output. In the following, we analyze this example in a mismatched decoding framework and obtain a decoding metric which minimizes the probability of error.

We study the probability that the decoder outputs a message different from the one sent, i.e., $\mathbb{P}\left[\hat{W} \neq W\right]$ while using a given codebook $\mathcal{C} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$. Observing the output $\boldsymbol{y}$, the decoder outputs the message w with probability

$$P_{\hat{W}|\boldsymbol{Y}}(w|\boldsymbol{y}) = \begin{cases} \frac{1}{|\mathcal{S}|}, & \text{if } w \in \mathcal{S} \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where the set $\mathcal{S}$ is defined as $\mathcal{S} = \left\{w | q(\boldsymbol{x}(w), y_1^{n-1}) = \max_{w'} q(\boldsymbol{x}(w'), y_1^{n-1})\right\}$. It is clear that the probability $P_{\hat{W}|\boldsymbol{Y}}(w|\boldsymbol{y})$ is independent of the n-th symbol of the output y_n, i.e., $P_{\hat{W}|\boldsymbol{Y}}(w|\boldsymbol{y}) = P_{\hat{W}|Y_1^{n-1}}(w|y_1^{n-1})$. The average error probability of the decoder is given by

$$\begin{aligned}
&\mathbb{P}\left[\hat{W} \neq W\right] && (21)\\
&= 1 - \sum_{w,\boldsymbol{y}} P_{\boldsymbol{XY}}(\boldsymbol{x}(w), \boldsymbol{y}) P_{\hat{W}|Y_1^{n-1}}(w|y_1^{n-1}) && (22)\\
&= 1 - \sum_{w, y_1^{n-1}} \Big[P_{\{XY\}_1^{n-1}}(x_1^{n-1}(w), y_1^{n-1}) P_{\hat{W}|Y_1^{n-1}}(w|y_1^{n-1}) \\
&\qquad\qquad \times \sum_{y_n} P_{Y_n|X_n}(y_n|x_n(w)) \Big] && (23)\\
&= 1 - \sum_{w, y_1^{n-1}} P_{\{XY\}_1^{n-1}}(x_1^{n-1}(w), y_1^{n-1}) P_{\hat{W}|Y_1^{n-1}}(w|y_1^{n-1}) && (24)\\
&\geq 1 - \sum_{y_1^{n-1}} \max_{w'} P_{\{XY\}_1^{n-1}}(x_1^{n-1}(w'), y_1^{n-1}) && (25)\\
&= 1 - \sum_{w, y_1^{n-1}} P_{\{XY\}_1^{n-1}}(x_1^{n-1}(w), y_1^{n-1}) P^{\text{MAP}}_{\hat{W}|Y_1^{n-1}}(w|y_1^{n-1}), && (26)
\end{aligned}$$

where $\{XY\}_1^{n-1}$ is the short form for the $X_1^{n-1} Y_1^{n-1}$.

The analysis shows that the metric $q(\boldsymbol{x}(w), \boldsymbol{y}) = \prod_{i=1}^{n-1} P_{Y|X}(y_i|x_i)$ which is the MAP metric for the first $n-1$ indices is the best possible mismatched decoding metric.

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